

# Exact analysis of disentanglement for continuous variable systems and application to a two-body system at zero temperature in an arbitrary heat bath

G. W. Ford

*Department of Physics, University of Michigan, Ann Arbor, MI 48109-1040 USA*

R. F. O'Connell\*

*Department of Physics and Astronomy,*

*Louisiana State University, Baton Rouge, LA 70803-4001 USA*

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## Abstract

We outline an exact approach to decoherence and entanglement problems for continuous variable systems. The method is based on a construction of quantum distribution functions introduced by Ford and Lewis [1] in which a system in thermal equilibrium is placed in an initial state by a measurement and then sampled by subsequent measurements. With the Langevin equation describing quantum Brownian motion, this method has proved to be a powerful tool for discussing such problems. After reviewing our previous work on decoherence and our recent work on disentanglement, we apply the method to the problem of a pair of particles in a correlated Gaussian state. The initial state and its time development are explicitly exhibited. For a single relaxation time bath at zero temperature exact numerical results are given. The criterion of Duan et al. [2] for such states is used to prove that the state is initially entangled and becomes separable after a finite time (entanglement sudden death).

Keywords: Disentanglement; Heat Bath; quantum Langevin equation; non-Markovian; Quantum Brownian motion

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\* Corresponding Author  
E-mail: oconnell@phys.lsu.edu

## I. INTRODUCTION

Simple quantum systems do not exist in isolation but are subject to environmental effects which can be simple temperature (in the case of negligible dissipation) and quantum effects or, more generally, also dissipative effects. The first quantitative treatment of such effects goes back to the phenomenological equations of Bloch for the description of nuclear magnetic resonance, with the well-known relaxation times  $T_1$  and  $T_2$ , which were later shown to arise from a solution of the master equation for a two-level system (describing a spin 1/2 system).

There has been much interest in recent years in small quantum systems, particularly those in quantum superposition states, which are generally entangled. Thus, in particular, for a single quantum particle in a superposition state in a heat bath, very short decoherence times arise. In order to calculate such times accurately, many different techniques have been proposed such as the Feynman-Vernon functional integral approach, the use of master equations and various stochastic methods. We have found that the quantum Langevin equation, supplemented by use of the Wigner distribution function, provides the basis of a powerful and physically transparent approach to such problems. In addition, our techniques are generally exact and lend themselves naturally to the incorporation of what we regard as the correct initial conditions. Moreover, memory and non-Markovian effects are naturally incorporated [3].

First, we want to lay to rest the notion that there is a useful master equation. We commence by examining initial conditions within the framework of master equations where one starts with an initially uncoupled quantum state, a free particle, say. Thus, the free particle is essentially at zero temperature with no cognizance of even the zero-point oscillations of the electromagnetic field. In addition, the initial state of the heat bath is in equilibrium at some temperature  $T$  but not coupled to the free particle. Next, the free particle and heat bath are brought into contact and, as we have shown explicitly [4], the free particle receives an initial impulse with the result that the center of the wave packet drifts to the origin. But, since for a free particle the origin cannot be a special point, we see that the translational invariance of the problem is broken by the assumption that the initial state corresponds to an uncoupled system. This problem exists in so-called "exact" master equation formulations, which are exact only in the sense that they incorporate time-dependent coefficients but they suffer from the same defects as the more conventional master equations; in fact,

the same results arise more easily from the use of the initial value Langevin equation which enabled us to obtain solutions of these "exact" master equations in a much more simplified form than one finds in the literature [4].

The problem with choosing an initial state corresponding to a particle at temperature  $T = 0$  does not give problems with nuclear magnetic resonance calculations where one deals with relatively long times, in contrast to the short times involved in decoherence and disentanglement calculations. However, even for the latter cases, we have shown [4] that one could circumvent this problem by choosing an initial corresponding to a wave packet at temperature  $T$  (obtained by averaging the initial Wigner function over a thermal distribution of initial velocities); as a result, the variance for very short times includes the thermal spreading but the initial impulse, resulting from bringing the quantum particle into contact with the heat bath, still remains. The end result is that " - - worthwhile results [for the exact master equation] can only be obtained in the high temperature limit" but, as we showed in a follow-up paper, in addition to irremedial divergencies due to zero-point fluctuations arising with exact master equations at low temperatures, in the high temperature regime (where, by convention, zero point fluctuations are neglected), problems also exist, notably the fact that the density matrix is not necessarily positive [5]. Moreover, in earlier work [6], Karrlein and Grabert showed in general that " - - there is no Liouville operator independent of the initial preparation - - .", which " - - is intimately connected with the failure of the Onsager regression hypothesis [7] in the quantum regime." In other words, there is no unique master equation.

Turning now to our quantum Langevin equation approach [3], which stemmed from the special FKM model [8], we considered a quantum particle coupled to a linear passive heat bath. In the distant past the quantum particle and heat bath are assumed to be in thermal equilibrium. Thus, we start with a complete system that is entangled with the bath at all times. At an initial time a "measurement" is made which prepares the system in an initial state. Then, at a later time, a second measurement is made which samples the state at that time. This formulation was explained in detail with a number of applications in a long paper in Phys. Rev. A [9], to which we refer the reader for further detail. In essence, we have extended the work Ford and Lewis [1] which itself is a quantum extension of the work of Wang-Uhlenbeck and Kolomogorov on joint probability distributions describing a classical stochastic process. In particular, we showed how the prescription can be extended in the form of a general formula for the Wigner function of a Brownian particle entangled

with a heat bath. The Wigner function provides the same information as the corresponding density matrix while making the calculations simpler and more transparent. This enabled us to calculate decoherence times for a variety of physical systems.

Entanglement is a subject of much current interest because of its key role in most applications in quantum information systems [10, 11]. Thus, its possible loss due to "entanglement sudden death" at a finite time has led to widespread interest in investigating this phenomenon [12]. The *bête noire* of entangled systems is the presence of a heat bath and its effect has generally been investigated using master equation techniques, despite their inherent limitations [5, 6, 9], as we have already pointed out. However, for the case of entangled continuous variable states, an exact analysis is possible, as we will now show. In particular, such systems are of interest in connection with linear optical quantum computing.

We consider two particles, each of mass  $m$ , at positions  $x_1$  and  $x_2$  in an initially entangled Gaussian state. In the absence of a heat bath, we already showed that this proved to be a very useful system for judging results from the use of various entanglement measures [13]. As we will now show, entanglement sudden death even occurs at zero temperatures. Here, we use an exact general prescription for treating both decoherence decay and entanglement decay for a broad class of entangled systems, in an arbitrary heat bath, just as we did for the decay of coherence of a single quantum system [15].

For a two-particle entangled system in a heat bath, the procedure is a straightforward generalization of the method described in [9] for the case of a single particle in an arbitrary heat bath so that, instead of starting with an initial state described by a single particle Gaussian wave function, we start with a two-particle Gaussian wave function. Then, in order to test for separability we use the Duan et al. criterion for such Gaussian states[2]. In Sec. II, we present our calculation and our conclusions are discussed in Sec. III.

## II. GAUSSIAN STATE FOR AN ENTANGLED TWO-PARTICLE SYSTEM

The Wigner characteristic function (the Fourier transform of the Wigner function) is given by the obvious generalization of Eq. (6.5) of [9]

$$\tilde{W}(Q_1, P_1; Q_2, P_2; t) = \frac{\langle f^\dagger(1) e^{-i(x_1(t)P_1 + m\dot{x}_1(t)Q_1 + x_2(t)P_2 + m\dot{x}_2(t)Q_2)/\hbar} f(1) \rangle}{\langle f^\dagger(1)f(1) \rangle}, \quad (1)$$

where the initial measurement is described by

$$f(1) = f(x_1(0) - x_1, x_2(0) - x_2). \quad (2)$$

in which  $f(x_1, x_2)$  is the  $c$ -number function describing the initial measurement where  $x_1(t)$  and  $x_2(t)$  are the time-dependent Heisenberg operators corresponding to the displacement of either particle:

$$x_j(t) = e^{iHt/\hbar} x_j(0) e^{-iHt/\hbar} \quad (3)$$

and the brackets indicate expectation with respect to the state of the system in equilibrium at temperature  $T$ .

In order to evaluate this formula we make the key assumption that particles are linear oscillators coupled to a linear passive heat bath and that within the bath the particles are *widely separated* so that we may ignore bath-induced interactions (a requirement imposed by most investigators, for example [16]). Thus,  $x_1(t)$  and  $x_2(t)$  independently undergo quantum Brownian motion. If we repeat the discussion leading to Eq. (6.43) of [9] we obtain

$$\begin{aligned} & \langle f^\dagger(1) e^{-i(x_1(t)P_1 + m\dot{x}_1(t)Q_1 + x_2(t)P_2 + m\dot{x}_2(t)Q_2)/\hbar} f(1) \rangle \\ &= \exp\left\{-\sum_{n=1}^2 \frac{\langle x^2 \rangle (P_n^2 - K_n^2) + m^2 \langle \dot{x}^2 \rangle Q_n^2}{2\hbar^2}\right\} \\ & \times \int_{-\infty}^{\infty} dx'_1 \int_{-\infty}^{\infty} dx'_2 f^\dagger\left(x'_1 + \frac{L_1}{2}, x'_2 + \frac{L_2}{2}\right) f\left(x'_1 - \frac{L_1}{2}, x'_2 - \frac{L_2}{2}\right) \\ & \times \frac{1}{2\pi \sqrt{\langle x^2 \rangle \langle \dot{x}^2 \rangle}} \exp\left\{-\sum_{n=1}^2 \frac{(x_n + x'_n)^2}{2 \langle x^2 \rangle} - i(x_n + x'_n) \frac{K_n}{\hbar}\right\}, \end{aligned} \quad (4)$$

where  $\langle x^2 \rangle$  and  $\langle \dot{x}^2 \rangle$  are the equilibrium variances for displacement and velocity, the same for each particle, and we have introduced

$$K_n = \frac{cP_n + m\dot{c}Q_n}{\langle x^2 \rangle}, \quad L_n = GP_n + m\dot{G}Q_n. \quad (5)$$

Here  $G = G(t)$  is the Green function where  $G(t) = [x(0), x(t)]/i\hbar$ . For explicit expressions suitable for numerical computation of these functions, see Appendix A of [9]. Also  $c = c(t) \equiv \frac{1}{2} \langle x(t)x(0) + x(0)x(t) \rangle$  is the correlation function, again the same for each particle.

These expressions are valid for any measurement function. We now specialize to the case where the initial measurement function is a Gaussian of the form

$$f(x_1, x_2) = \frac{(a_{11}a_{22} - a_{12}^2)^{1/4}}{\sqrt{2\pi}} \exp\left\{-\frac{a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2}{4}\right\}. \quad (6)$$

Then (4) becomes

$$\begin{aligned}
& \langle f^\dagger(1) e^{-i(x_1(t)P_1 + m\dot{x}_1(t)Q_1 + x_2(t)P_2 + m\dot{x}_2(t)Q_2)/\hbar} f(1) \rangle \\
&= \frac{\sqrt{a_{11}a_{22} - a_{12}^2}}{(2\pi)^2 \sqrt{\langle x^2 \rangle \langle x'^2 \rangle}} \exp\left\{-\frac{a_{11}L_1^2 + 2a_{12}L_1L_2 + a_{22}L_2^2}{8}\right\} \\
&\quad \times \exp\left\{-\sum_{n=1}^2 \frac{\langle x^2 \rangle (P_n^2 - K_n^2) + m^2 \langle \dot{x}^2 \rangle Q_n^2}{2\hbar^2}\right\} \\
&\quad \times \int_{-\infty}^{\infty} dx'_1 \int_{-\infty}^{\infty} dx'_2 \exp\left\{-\frac{a_{11}x_1'^2 + 2a_{12}x'_1x'_2 + a_{22}x_2'^2}{2}\right\} \\
&\quad \times \exp\left\{-\sum_{n=1}^2 \frac{(x_n + x'_n)^2}{2\langle x^2 \rangle} - i(x_n + x'_n) \frac{K_n}{\hbar}\right\}. \tag{7}
\end{aligned}$$

The integral is standard Gaussian and we find

$$\begin{aligned}
& \langle f^\dagger(1) e^{-i(x_1(t)P_1 + m\dot{x}_1(t)Q_1 + x_2(t)P_2 + m\dot{x}_2(t)Q_2)/\hbar} f(1) \rangle \\
&= \frac{\sqrt{a_{11}a_{22} - a_{12}^2}}{2\pi \sqrt{(a_{11}\langle x^2 \rangle + 1)(a_{22}\langle x^2 \rangle + 1) - a_{12}^2\langle x^2 \rangle \langle x'^2 \rangle}} \\
&\quad \times \exp\left\{-\frac{a_{11}L_1^2 + 2a_{12}L_1L_2 + a_{22}L_2^2}{8}\right\} \\
&\quad \times \exp\left\{-\sum_{n=1}^2 \frac{\langle x^2 \rangle (P_n^2 - K_n^2) + m^2 \langle \dot{x}^2 \rangle Q_n^2}{2\hbar^2}\right\} \\
&\quad \times \exp\left\{-\frac{(a_{22} + \langle x^2 \rangle^{-1})K_1^2 - 2a_{12}K_1K_2 + (a_{11} + \langle x^2 \rangle^{-1})K_2^2}{2\hbar^2[(a_{11} + \langle x^2 \rangle^{-1})(a_{22} + \langle x^2 \rangle^{-1}) - a_{12}^2]}\right\}, \tag{8}
\end{aligned}$$

where we have chosen  $x_1 = x_2 = 0$  in order that the initial state be centered at the origin.

The Wigner characteristic function is

$$\begin{aligned}
& \tilde{W}(Q_1, P_1; Q_2, P_2; t) \\
&= \exp\left\{-\frac{a_{11}L_1^2 + 2a_{12}L_1L_2 + a_{22}L_2^2}{8}\right\} \\
&\quad \times \exp\left\{-\sum_{n=1}^2 \frac{\langle x^2 \rangle (P_n^2 - K_n^2) + m^2 \langle \dot{x}^2 \rangle Q_n^2}{2\hbar^2}\right\} \\
&\quad \times \exp\left\{-\frac{(a_{22} + \langle x^2 \rangle^{-1})K_1^2 - 2a_{12}K_1K_2 + (a_{11} + \langle x^2 \rangle^{-1})K_2^2}{2\hbar^2[(a_{11} + \langle x^2 \rangle^{-1})(a_{22} + \langle x^2 \rangle^{-1}) - a_{12}^2]}\right\}, \tag{9}
\end{aligned}$$

This becomes simpler in the free particle limit :  $\langle x^2 \rangle \rightarrow \infty$ . Noting that near the center of an oscillator potential the motion is that of a free particle so that, in this limit, the measurement function [6] is, in essence, the wave function for the initial state of the free particle. In essence,  $\langle x^2 \rangle \rightarrow \infty$  corresponds to the range of the oscillator getting larger or,

concomitantly, the oscillator becomes so weak as to be indistinguishable from that of a free particle. Thus, in this limit

$$\begin{aligned}\tilde{W}(Q_1, P_1; Q_2, P_2; t) = & \exp\left\{-\frac{a_{11}L_1^2 + 2a_{12}L_1L_2 + a_{22}L_2^2}{8}\right\} \\ & \times \exp\left\{-\frac{a_{22}P_1^2 - 2a_{12}P_1P_2 + a_{11}P_2^2}{2\hbar^2(a_{11}a_{22} - a_{12}^2)}\right\} \\ & \times \exp\left\{-\sum_{n=1}^2 \frac{sP_n^2 + m\dot{s}P_nQ_n + m^2\langle\dot{x}^2\rangle Q_n^2}{2\hbar^2}\right\},\end{aligned}\quad (10)$$

where  $s = s(t) = \langle(x(t) - x(0))^2\rangle = 2\langle x^2\rangle - 2c(t)$  is the mean square displacement.

Then, in particular, we find the initial state of the Wigner characteristic function to be

$$\begin{aligned}\tilde{W}(Q_1, P_1; Q_2, P_2; 0) = & \exp\left\{-\frac{a_{11}Q_1^2 + 2a_{12}Q_1Q_2 + a_{22}Q_2^2}{8}\right\} \\ & \times \exp\left\{-\frac{a_{22}P_1^2 - 2a_{12}P_1P_2 + a_{11}P_2^2}{2\hbar^2(a_{11}a_{22} - a_{12}^2)}\right\} \\ & \times \exp\left\{-\sum_{n=1}^2 \frac{m^2\langle\dot{x}^2\rangle Q_n^2}{2\hbar^2}\right\}.\end{aligned}\quad (11)$$

For simplicity, we will henceforth confine ourselves to the symmetric case where  $a_{22} = a_{11}$ . In this case we can write the Wigner characteristic function in the form

$$\tilde{W}(P_1, Q_1, P_2, Q_2) = \exp\left\{-\frac{1}{2}\mathbf{X} \cdot \mathbf{M} \cdot \mathbf{X}\right\}, \quad (12)$$

where

$$\mathbf{X} = \begin{pmatrix} \frac{LP_1}{\hbar} \\ \frac{Q_2}{\hbar} \\ \frac{LP_2}{\hbar} \\ \frac{Q_2}{L} \end{pmatrix}, \quad \mathbf{M} = \begin{pmatrix} G_{11} & G_{12} & C_{11} & C_{12} \\ G_{12} & G_{22} & C_{21} & C_{22} \\ C_{11} & C_{21} & G_{11} & G_{12} \\ C_{12} & C_{22} & G_{12} & G_{22} \end{pmatrix} = \begin{pmatrix} \mathbf{G} & \mathbf{C} \\ \mathbf{C} & \mathbf{G} \end{pmatrix}. \quad (13)$$

Here  $L$  is a constant of dimension length, introduced to make the matrix elements of the correlation matrix  $\mathbf{M}$  and the elements of  $\mathbf{X}$  dimensionless. For the state with the Wigner

characteristic function given by the symmetric limit of (10), we find that

$$\begin{aligned}
G_{11} &= \frac{1}{L^2} \left[ \frac{a_{11}}{a_{11}^2 - a_{12}^2} + \left( \frac{\hbar G}{2} \right)^2 a_{11} + s \right], \\
G_{12} &= \left( \frac{\hbar G}{2} \right) \left( \frac{m\dot{G}}{2} \right) a_{11} + \frac{m\dot{s}}{2\hbar}, \\
G_{22} &= L^2 \left[ \frac{m^2 \langle \dot{x}^2 \rangle}{\hbar^2} + \left( \frac{m\dot{G}}{2} \right)^2 a_{11} \right], \\
C_{11} &= \frac{1}{L^2} \left[ -\frac{a_{12}}{a_{11}^2 - a_{12}^2} + \left( \frac{\hbar G}{2} \right)^2 a_{12} \right], \\
C_{12} &= C_{21} = \left( \frac{\hbar G}{2} \right) \left( \frac{m\dot{G}}{2} \right) a_{12}, \\
C_{22} &= L^2 \left( \frac{m\dot{G}}{2} \right)^2 a_{12}.
\end{aligned} \tag{14}$$

In these expressions we recall that  $G$  is the Green function and  $s = \langle (x(t) - x(0))^2 \rangle$  is the mean square displacement, the same for both particles [9].

In order to discuss entanglement, Duan et al. [2] perform a sequence of rotations and squeezes to bring  $\mathbf{M}$  to a form in which

$$\mathbf{G} = \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} c & 0 \\ 0 & c' \end{pmatrix}. \tag{15}$$

Since the determinants are invariant under these transformations, we have the following simple relations for determining the quantities  $g$ ,  $c$  and  $c'$  in terms of these invariants

$$\det \mathbf{G} = g^2, \quad \det \mathbf{C} = cc', \quad \det \mathbf{M} = (g^2 - c^2) (g^2 - c'^2). \tag{16}$$

The necessary and sufficient condition that the state be disentangled is equivalent to the inequality

$$\sqrt{(g - c)(g + c')} \geq \frac{1}{2}. \tag{17}$$

This result is equivalent to that obtained by the Duan et al. analysis, specialized to the symmetric case.

We have calculated these quantities for the case of two particles coupled to a single-relaxation time bath [9] at *zero temperature*. This heat bath is characterized by a memory function of the form

$$\mu(t) = \frac{\zeta}{\tau} e^{-t/\tau} \theta(t), \tag{18}$$



in the quantum Langevin equation [3]. Here  $\theta(t)$  is the Heaviside function and we note that in the limit  $\tau \rightarrow 0$  this becomes the Ohmic memory function  $\mu(t) = 2\zeta\delta(t)$ . The corresponding Fourier transform of (18) is

$$\tilde{\mu}(\omega) = \frac{\zeta}{1 - i\omega\tau} \quad (19)$$

whose Ohmic limit is  $\zeta \equiv m\gamma$ .

Defining

$$b_{ij} = \frac{\zeta}{\hbar} a_{ij} \quad (20)$$

which are dimensionless constants proportional to the  $a_{ij}$ , we choose  $L^2 = \hbar/\zeta$  and  $\gamma/\tau = \frac{1}{5}$  for two different selections of the  $b_{ij}$  quantities. Thus, the choice  $b_{11} = 5$ ,  $b_{12} = 4$  implies that the ratio  $(a_{12}/a_{11}) = \frac{4}{5}$  whereas the choice  $b_{11} = 5,000$ ,  $b_{12} = 4,999$  implies that  $(a_{12}/a_{11}) = 0.9998$ . Thus, as expected, the latter choice corresponding to relatively larger  $a_{12}$ , encounters sudden death at a later time. In Fig. 1, we plot the left hand side of the above Duan inequality versus  $\gamma t$ . Note that the curve crosses 0.5 (signifying entanglement sudden death) at two different  $\gamma t$  values. Thus, we see exactly that entanglement sudden death occurs later for larger  $a_{12}/a_{11}$  values.

### III. CONCLUSIONS

The generic problem of a quantum system in an environment (heat bath) has been tackled by two main approaches, the Feynman-Vernon approach, with the use of master equations, and the quantum Langevin equation approach. We have argued that the latter method is generally superior as it treats the whole system as being completely entangled in thermal equilibrium to begin with.

How the system attains thermal equilibrium is often referred to as the zeroth law of thermodynamics [17], which goes back to the fundamental ideas of Boltzmann and Gibbs. In essence, the microscopic laws are time-reversal invariant and the Poincaré recurrence theorem seems to preclude the achievement of equilibrium. However, the latter can be achieved by recognizing that thermal equilibrium is a macroscopic notion and that the relaxation to equilibrium depends on coarse graining and also the Hamiltonian [17]. In practice, as we have done in our initial paper on the quantum Langevin equation [3] and later in [4], in our discussion of the inhomogeneous equation (see section IV of [3]), we have chosen the retarded

solution, thereby breaking the time-reversal invariance of the original equations. This could be achieved, for instance, by fastening the quantum particle to a large mass in the distant past so that it is held fixed at  $x = 0$  say with zero momentum. The large (eventually infinite) number of oscillators are then allowed to come to equilibrium at temperature  $T$ , say, by weak interaction with another bath (similar to how a collection of particles in a container come to equilibrium by interacting with the walls of the container). Then, still in the distant past, the system is released and the subsequent motion is governed by the appropriate Hamiltonian. As we concluded in [3], this "is typical of the way time-reversal invariance is broken in macroscopic systems: they describe only the time development of a class of solutions of the microscopic equations." The end result is that at  $t = 0$ , say, our complete system is in thermal equilibrium at temperature  $T$ . The system then develops unitarily in time after which we prepare the system in a desired state by means of a first measurement on the quantum particle. Then, at a later time, we do a second measurement which tells us how the system has developed in time due to environmental effects. If desired, subsequent measurements can be carried out in a similar manner. It should be again emphasized that our procedure is exact.

Our method applies to arbitrary heat baths and arbitrary temperatures. In particular, calculations at zero temperature are readily carried out [18] without encountering the irreducible divergences associated with exact master equations [4]. Having previously applied this method to a variety of decoherence problems (involving a single particle prepared in a variety of initial states and analyzing its subsequent development in a arbitrary heat bath at arbitrary temperatures), we concluded by writing a detailed paper [9], showing in particular that our work constitutes a quantum extension of the classical stochastic theory on joint probability distributions.

Next, we turned to the disentanglement problem which involves more than one particle. In particular, for the two-particle system, we commenced by showing that our general techniques were very useful for judging disentanglement in the absence of a heat bath [13]. Next, in our present paper, we have extended our work in [9] to the case of two particles in an initial correlated Gaussian state and we studied the time development of this state in an arbitrary heat bath at zero temperature. We found that the state which is initially entangled becomes separable after a finite time. Thus, entanglement sudden death is also prevalent in continuous variable systems, as well as the more often studied qubit systems

[12], which should raise concern for the designers of all entangled systems. A key question is the dependence of the sudden death on the number of particles. The procedure is again a generalization of the method described in [9] but we expect that the computational task will be formidable, unless we discover some creative approaches.

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FIG. 1: The left side of the Duan inequality (17) as a function of time  $\gamma t$ , for two values of the parameters appearing in the initial wave function (6). The horizontal dotted line signifies where entanglement sudden death occurs.